American Journal of Alternative Education
ISSN: 2997-3953 (Online)
Vol. 2, No. 12, Dec 2025,
Available online at: https://scientificbulletin.com/index.php/AJAE

On Iterations of One Complex Variable Functions

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Article information:
Manuscript received: 02 Oct 2025; **Accepted:** 02 Nov 2025; **Published:** 04 Dec 2025

Abstract: The concept of complex dynamic systems is one of the fundamental branch of complex analysis, which has been developing for many years. A lot of mathematicians have been engaged in complex dynamic systems, and also nowadays this field is expanding. The complex dynamic systems have many practical applications. Gaston Julia, Pierre Fatou as well as John Milnor and many other scientists had contributed to the development of this field.

In this article, certain properties of rational functions and iterations of some holomorphic functions are investigated. It is also proved that if the unit circle

$C = \{z \in \mathbb{C} \| z |= 1\}$ is forward invariant under the action of a polynomial $P_d(z)$ of degree $d \geq 2$, then $P_d(z) = \alpha z^d$, $\quad |\alpha| = 1$..

Keys words: holomorphic functions, iterations, conjugate function, forward invariant set, backward invariant, fixed point, attracting.

INTRODUCTION

The study of invariant sets under holomorphic maps represents a fundamental problem in complex dynamics. While the behavior of rational functions on general Julia sets has been extensively investigated, the special case of circle-preserving maps remains of particular interest due to its connections with geometric function theory and polynomial algebra.

This work addresses the classification of polynomials that leave the unit circle invariant. Previous results in this direction have established partial characterizations, but a complete classification for arbitrary degree polynomials has remained open. Our main contribution resolves this problem by demonstrating that if a polynomial $P_d(z)$ of degree $d \geq 2$ satisfies $P_d(S^1) \subseteq S^1$, then it must necessarily be a monomial $P_d(z) = \alpha z^d$ with $|\alpha| = 1$..

The significance of this result is twofold: it completes the classification of circle-preserving polynomials, and it reveals the exceptional rigidity of such maps. Our proof employs elementary methods based on the linear independence of monomial functions, providing a transparent and accessible approach to this invariant set problem.

BRIEF ANALYSIS OF REFERENCES

The study of complex dynamical systems has been extensively developed in several landmark texts. Beardon's comprehensive treatment [1] establishes the fundamental framework for iterated rational maps, while Milnor's modern exposition [2] provides crucial insights into Fatou-Julia theory. The geometric perspective of Carleson and Gamelin [3] offers valuable approaches to invariant sets, and

Ahlfors' classical work [4] underpins the complex-analytic foundations.

MATERIALS AND METHODS

We denote the complex plane by $\mathbb{C}$ as usual. For many purposes it is useful to extend the system $\mathbb{C}$ of complex numbers by introducing symbol ∞ to represent infinity and we denote extended plane by $\overline{\mathbb{C}}$ $\left(\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}\right)$.

It is known [1], [4] that one can represent $\overline{\mathbb{C}}$ by stereographic protection of a unit sphere centered at the origin in $\sim^{3}$ i.e. which is called a Riemannian sphere. The Euclidean metric in the Riemann sphere induces the metric σ in $\overline{\mathbb{C}}$, where

$$\sigma(z,w)=\frac{2|z-w|}{\left(1+|z|^2\right)^{1/2}\left(1+|w|^2\right)^{1/2}},\ z,w\in\mathbb{C},$$

$$\sigma(z,\infty)=\frac{2}{\left(1+|z|^2\right)^{1/2}},\ z\in\mathbb{C}.$$

As a result $\overline{\mathbb{C}}$ will be a compact metric space.

A rational map is a function of the form $R(z)=\dfrac{P(z)}{Q(z)}$, where P and Q are polynomials, not both being the zero polynomial. If P is the zero polynomial, then R is the constant zero function, if Q is the zero polynomial, then R is the constant ∞ function (note that this is regarded as a rational function). If $Q(z)=0$ and P is not the zero polynomial, then $R(z)$ is defined to be ∞. Father we define $R(\infty)$ as the limit of $R(z)$ as $z\to\infty$.

Let P and Q be coprime polynomials. Degree $\deg(R)$ of $R(z)$ is defined by $\deg(R)=\max\{\deg(P),\deg(Q)\}$.

If $R(z)$ is a constant function, with value c, where $c\neq 0,\infty$, we have $\deg(R)=0$ and it is convenient to define $\deg(R)=0$ even when c is 0 or ∞.

Thereby rational functions are defined on $\overline{\mathbb{C}}$.

Definition 1. The function R is called holomorphic at ∞, if the function $g(z)=R\left(\dfrac{1}{z}\right)$ is holomorphic at $z=0$.

But we need to remember that a derivative of R at infinity does not mean. $q\neq\infty$ is called ***poles*** of the function R, if $R(q)=\infty$. The function $z\to 1/R(z)$ is holomorphic near the poles q, then $z\to 1/R(z)$ equal to zero at q.

Now we will show that the rational function is holomorphic in $\overline{\mathbb{C}}$. It is clear the rational function is holomorphic in $\overline{\mathbb{C}}\setminus\left(\{z\in\mathbb{C}:Q(z)=0\}\cup\{\infty\}\right)$. Therefore we have to verify a function R for holomorphicity at the poles and ∞.

Suppose, $\deg(P)=n$, $\deg(Q)=m(n>m)$ and P, Q are coprime polynomials. In this case the function $R(z)$ equivalent to cz^{n-m} as $z\to\infty$ $(c\neq 0, c\in\mathbb{C})$. Such that $R(\infty)=\infty$. By the first definition, for to check the holomorphicity of R at the point ∞, we verify $\frac{1}{R(1/z)}$ for holomorphicity at the point $z=0$. thus $R^{-1}(1/z)=z^{n-m}/c$ is holomorphic at the point $z=0$. If the inequality $m>n$ holds, $R(z)$ is equivalent to $g(z)=\frac{c}{z^{m-n}}$ $(c\neq 0, c\in\mathbb{C})$ as

$z\to\infty$, then $R(\infty)=0$. Moreover, $R\left(\frac{1}{z}\right)$ is holomorphic at the point $z=0$ therefore $R(z)$ is holomorphic $z=\infty$ as $m>n$.

It is not necessary to verify for holomorphicity of $R(z)$ in the case $n=m$. We now check holomorphicity of the function $R(z)$ at the poles $q\in\mathbb{C}\,(Q(q)=0)$. In a sufficient small neighborhood of the poles $z=q$, the function $R(z)$ can be replaced with the function $\frac{c}{(z-a)^m}$ $(c\in\mathbb{C}, m>0)$.

In the similar way, for to check the holomorphicity of $R(z)$ at the point q, we verify holomorphicity of $\frac{1}{R(z)}$ at the point q. As a result the function $\frac{1}{R(z)}$ is holomorphic at the polis and $R(z)$ is holomorphic at the polis.

Thus the rational function $R:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is holomorphic on the $\overline{\mathbb{C}}$.

It is known that a holomorphic function $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is only rational function ([1:31], [2:41]).

Iterations of a holomorphic function

Suppose, $f:D\to D, \left(D\subset\overline{\mathbb{C}}\right)$ is a complex valued function. We call the sequence $f^n(z)=\underbrace{f\circ f\circ\ldots\circ f(z)}_{n}=f\left(f^{n-1}(z)\right), n=1,2,\ldots$ n^{th} ***order iteration*** of the function $f(z)$.

These iterations determine discrete dynamical system. Main goal of the theory of discrete dynamical systems is exploring behavior of the sequence $\left\{f^n(z)\right\}_{n=1}^{\infty}$ at various points $z\in D$.

Definition 2. A point $\zeta\in D$ is called fixed point of f if $f(\zeta)=\zeta$.

If the fixed point ζ of f lies in , then the derivative $f'(\zeta)$ is defined and we say that ζ is:

I. an attracting fixed point if $|f'(\zeta)|<1$;

II. a repelling fixed point if $|f'(\zeta)|>1$; and

III. an indifferent fixed point if $\left|f'(\zeta)\right| = 1$.

Example 1. Consider iterations of the function $R(z) = \dfrac{5z-4}{4z-3}$.

It is easy to verify, that

$$R(z) = \frac{5z-4}{4z-3},\ R^2(z) = R(R(z)) = \frac{5R(z)-4}{4R(z)-3} = 1 + \frac{z-1}{8z-7} \ldots,$$

$$R^n(z) = R\left(R^{n-1}(z)\right) = \frac{(4n+1)z-4n}{4nz-(4n-1)} = 1 + \frac{z-1}{4nz-(4n-1)}.$$

$R(z)$ has fixed point $z = 1$ and it is indifferent fixed point i.e. $\left|R'(z)\right| = 1$, but we can easyly verify that for any $z \in \mathbb{C}$ $\lim\limits_{n\to\infty} R^n(z) = 1$.

We say that a function $f: U \to U$ is conjugate to a function $g: V \to V$ if there is a conformal map $\varphi: U \to V$ such that $g = \varphi \circ f \circ \varphi^{-1}$,

that is $\varphi(f(z)) = g(\varphi(z))$.

The maps f and g can be regarded as the same map viewed in different coordinate systems. The definition implies the iterates f^n and g^n are also conjugate, $g^n = \varphi \circ f^n \circ \varphi^{-1}$, as are f^{-1} and g^{-1} when defined, $g^{-1} = \varphi \circ f^{-1} \circ \varphi^{-1}$.Note that φ maps fixed points of f to fixed points of g, and the multipliers at the corresponding fixed points are equal. A basin of attraction for f is mapped by a conjugating map φ onto a basin [2] of attraction for g.

Definition 3. Let $f: D \to D$ is a holomorphic function. The family $\left\{f^n(z)\right\}$ in a domain D is said to be normal if it contains either a subsequence that converges uniformly on every compact set $K \subset D$, or a subsequence that tends uniformly to ∞ on every compact set.

Definition 4. The Fatou set $F(f)$ of f is defined to be the set of points $z_0 \in \overline{\mathbb{C}}$ such that $\left\{f^n(z)\right\}$ is a normal family in some neighborhood of z_0. The Julia set $J(f)$ is the complement of the $F(f)$.

Example 2. Let us find Fatou and Julia sets of $p(z) = z^2 - 2$.

One can easily verify $p(z) = z^2 - 2$ is conjugated to the Tchebychev polynomial $g(z) = 2z^2 - 1$ by $\varphi(z) = 2z$, i.e. $p(z) = \varphi \circ g \circ \varphi^{-1}(z)$. It is known

$([1:9])$, that $J(g) = [-1,1]$ and $F(g) = \overline{\mathbb{C}} \setminus [-1,1]$ then if $z \in \overline{\mathbb{C}} \setminus [-1,1]$, $\lim\limits_{n\to\infty} g^n(z) = \infty$.

Moreover $\varphi([-1,1]) = [-2,2]$, therefore the Julia set of $p(z) = z^2 - 2$ is the closed interval $[-2,2]$, if $z \in \overline{\mathbb{C}} \setminus [-2,2]$ then $\lim\limits_{n\to\infty} p^n(z) = \infty$, such that $F(p) = \overline{\mathbb{C}} \setminus [-2,2]$.

Definition 5. A set $E \subset \overline{\mathbb{C}}$ is called forward invariant if $f(E) \subset E$

Definition 6. A set $E \subset \overline{\mathbb{C}}$ is called backward invariant if $f^{-1}\left(E\right) \subset E$

DISCUSSION AND RESULTS

Theorem: Let $P_d\left(z\right)$ be a polynomial of degree at least two $\left(d \geq 2\right)$ and

that the unit circle centered at the origin C is forward invariant. Then $P_d(z) = \alpha z^d \left(\left|\alpha\right| = 1\right)$.

Suppose we are given the circle C and a polynomial $P_d(z)$. Let the circle C be forward invariant.

$P_d(z) = a_d z^d + a_{d-1} z^{d-1} + \ldots + a_0,\ a_d \neq 0$. Now we will show that $P_d(z) = \alpha z^d \left(\left|\alpha\right| = 1\right)$.

It is obvious that if $z \in C$ then $\left|P_d(z)\right| = 1$. Thus $P_d(z) \cdot \overline{P_d(z)} = 1$. It is known if $z \in C$, $z = e^{i\varphi}\left(\varphi \in \left[0, 2\pi\right]\right)$, then we obtain the following equation

$$P_d(e^{i\varphi}) \cdot \overline{P_d(e^{i\varphi})} - 1 = \left(\left|a_0\right|^2 + \left|a_1\right|^2 + \ldots + \left|a_{d-1}\right|^2 + \left|a_d\right|^2 - 1\right) \cdot 1 + \sum_{i=0}^{d} (a_i \overline{a}_{i+1} + \overline{a}_i a_{i+1}) \cdot \cos(\varphi) +$$

$$+ \sum_{i=0}^{d} (a_i \overline{a}_{i+2} + \overline{a}_i a_{i+2}) \cdot \cos(2\varphi) + \ldots + (a_d \overline{a}_0 + \overline{a}_d a_0) \cdot \cos(d\varphi) +$$

$$+ \sum_{i=0}^{d} (a_i \overline{a}_{i+1} - \overline{a}_i a_{i+1}) \sin(\varphi) + \sum_{i=0}^{d} (a_i \overline{a}_{i+2} - \overline{a}_i a_{i+2}) \sin(2\varphi) + \ldots +$$

$$+ (a_d \overline{a}_0 - \overline{a}_d a_0) \sin(d\varphi) = 0.$$

It is known that the functions $1, \cos(\varphi), \cos(2\varphi), \ldots, \cos(d\varphi), \sin(\varphi), \sin(2\varphi), \ldots, \sin(d\varphi)$ are linear independent. Therefore

$$\left|a_0\right|^2 + \left|a_1\right|^2 + \ldots + \left|a_{d-1}\right|^2 + \left|a_d\right|^2 - 1 = 0,\ \sum_{i=0}^{d} (a_i \overline{a}_{i+1} + \overline{a}_i a_{i+1}) = 0,$$

$$\sum_{i=0}^{d} (a_i \overline{a}_{i+1} - \overline{a}_i a_{i+1}) = 0,\ \sum_{i=0}^{d} (a_i \overline{a}_{i+2} + \overline{a}_i a_{i+2}) = 0,\ \sum_{i=0}^{d} (a_i \overline{a}_{i+2} - \overline{a}_i a_{i+2}) = 0, \ldots,$$

$$a_d \overline{a}_0 + \overline{a}_d a_0 = 0,\ a_d \overline{a}_0 - \overline{a}_d a_0 = 0.$$

We obtain from the last two equations that $a_0 = 0$. Therefore $P_d(z) = a_d z^d + \ldots + a_1 z = z\left(a_d z^{d-1} + \ldots + a_1\right) = z P_{d-1}(z)$.

From the last equality it follows that $\left|P_d(z)\right| = \left|z\right|\left|P_{d-1}(z)\right| = \left|P_{d-1}(z)\right| = 1$,

where $P_{d-1}(z) = a_d z^{d-1} + \ldots + a_1$. Thus we obtain from the identity $P_{d-1}(z) \cdot \overline{P_{d-1}(z)} = 1$ that $a_1 = 0$. By the similar way we find that

$a_2 = a_3 = \ldots = a_{d-2} = a_{d-1} = 0$ and $\left|a_d\right| = 1$. Thus $P_d(z) = \alpha z^d$, $\left|\alpha\right| = 1$ as we promised.

This theorem generalizes theorem 1.3.1 of the [1].

Corollary. If the unit circle is forward invariant under the polynomial $P_d(z)$, then it is also backward invariant under the $P_d(z)$.

CONCLUSION

This paper has established a complete characterization of polynomials that preserve the unit circle under iteration. We have proven that if a polynomial $P_d(z)$ of degree $d \geq 2$ satisfies the forward invariance condition $P_d(C) \subset C$, then it must necessarily be a monomial of the form $P_d(z) = \alpha z^d$ with $|\alpha| = 1$.

This result demonstrates the exceptional rigidity of circle-preserving polynomial maps and extends previous partial characterizations found in the literature. The proof technique, based on the linear independence of monomial functions, provides an elementary yet powerful approach to solving invariance problems in complex dynamics.

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