SemantJournals American Journal of Education and Evaluation Studies 2997-9439, Vol 3, No.5, (May, 2026)

Article

Limit Theorems for Critical and Subcritical Branching Processes with Immigration

Sarvar Iskandarov
Urgench State University named after Abu Rayhan Biruni

Laylo Karimova
Urgench State University named after Abu Rayhan Biruni

Abstract: In this paper, we investigate limit theorems for critical and subcritical branching processes with immigration. We investigate the classical Yaglom-type conditional limit theorem and establish new asymptotic results under appropriate normalization. The presence of immigration leads to limiting behavior, transforming exponential limits into mote general distributions.

Keywords: Branching Processes, Critical and Subcritical Processes, Immigration in Stochastic Models

Introduction

Branching processes constitute an important class of stochastic models in probability theory and have been extensively investigated due to their applications in population dynamics, biology, and other applied fields. The classical Galton-Watson branching process provides a fundamental probabilistic framework for describing the evolution of populations with independent reproduction mechanisms [1], [2], [3].

A central role in the theory is played by the *critical case,* when the mean number of offspring satisfies

$$EX = 1 \quad (1)$$

In this regime, the behavior of the process is particularly delicate and has been studied in detail in numerous works (see, for example, [4,5]). One of the most important results in this direction is the *Yaglom-type conditional limit theorem,*first established in [6] and further developed in [7,8]. More precisely, if Z_n denotes the population size at generation $n,$ then

$$\frac{Z_n}{Bn} | \{Z_n > 0\} \to Exp(1), \quad (2)$$

where the constant B is determined by the second moment of the offspring distribution.

However, in many realistic models, the assumption of a closed population is not adequate, and it becomes necessary to incorporate *immigration* into the model [9]. Branching processes with immigration have been investigated by various authors (see [9], [10], [11]).

In particular, it is known that

$$n^{\lambda} P_n(s) \to \pi(s), \quad (3)$$

where $\pi(s)$ is a limiting generating function [12], [13].

Despite the extensive literature, the connection between classical conditional limit theorems and immigration models remains insufficiently explored [14].

Contribution of this paper. We extend classical results and derive new limit theorems for branching processes with immigration, showing a transition from exponential to more general limiting distributions.

Methodology

2. Preliminaries. In this section, we recall basic definitions and known results related to branching processes that will be used throughout the paper.

Galton-Watson branching process. Let $\{Z_n\}_{n\geq 0}$ be a Galton-Watson branching process defined by

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}, n \geq 0 \quad (2.1)$$

Where $\{X_{n,i}\}$ are independent and identically distributed non-negative integer-valued random variables [15].

Let the offspring generating function be defined as

$$F(s) = E[s^X] = \sum_{k=0}^{\infty} p_k s^k, 0 \leq s \leq 1, \quad (2.2)$$

Where $p_k = P(X = k)$.

The mean number of offspring is given by

$$m = F'(1) = E[X]. \quad (2.3)$$

The process is called *critical* if

$$m = 1. \quad (2.4)$$

Assume further that the second moment is finite:

$$F''(1) = 2B \in (0, \infty) \quad (2.5)$$

Let

$$Q_n = P(Z_n > 0) \quad (2.6)$$

denote the survival probability.

A classical result (Yaglom-type theorem) states that

$$Q_n \sim \frac{1}{Bn}, n \to \infty, \quad (2.7)$$

and

$$\lim_{n\to\infty} E[\exp\{-\lambda \frac{Z_n}{Bn}\} | Z_n > 0] = \frac{1}{1+\lambda}, \lambda \geq 0, \quad (2.8)$$

See.

This implies the conditional convergence

$$\frac{Z_n}{Bn} | \{Z_n > 0\} \to Exp(1). \quad (2.9)$$

Branching process with immigration. We now consider a branching process with immigration defined by

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + Y_n, \quad (2.10)$$

where $\{Y_n\}$ are i.i.d. non-negative integer-valued random variables.

Let the generating function of immigration be

$$G(s) = E[s^Y], 0 \leq s \leq 1 \quad (2.11)$$

Denote

$$\alpha = G'(1) = E[Y]. \quad (2.12)$$

Let $F_n(s)$be the n-th iterate of F(s) i.e.

$$F_0(s) = s, \quad u > 0. \quad (2.13)$$

The generating function of transition probabilities is given by

$$P_n(s) = E[s^{Z_n} | Z_0 = 0] = \prod_{k=0}^{n-1} G(F_k(s)), \quad (2.14)$$

See.

Assume that

$$\lambda = \frac{\alpha}{B}. \quad (2.15)$$

A known asymptotic result states that

$$n^\lambda P_n(s) \to \pi(s), n \to \infty, \quad (2.16)$$

Where $\pi(s)$,is a limiting generating function.

Furthermore, the refined asymptotic expansion

$$n^\lambda P_n(s) = \pi(s)(1 + \Delta \frac{\ln b_n(s)}{b_n(s)}(1 + o(1))) \quad (2.17)$$

holds [12], where

$$b_n(s) = Bn + \frac{1}{1-s'} \quad (2.18)$$

And

$$\Delta = \alpha(\frac{C}{6B^2} - 1). \quad (2.19)$$

Result and Discussion

3.1 Limit theorems for branching processes with immigration. In this section, we present limit theorems for critical branching processes with immigration. We begin with a known asymptotic result for generating functions.

Theorem 3.1. Assume that

$$F'(1) = 1,\ F''(1) = 2\ B \in (0, \infty) \quad (4)$$

and

$$G'(1) = \alpha \in (0, \infty) \quad (5)$$

Let

$$\lambda = \frac{\alpha}{B} \quad (6)$$

Then, for $0 \le s \le 1$,

$$n^{\lambda} P_n(s) \to \pi(s), n \to \infty, \quad (7)$$

where $\pi(s)$, is a limiting generating function [16].

We now present the main result of this paper.

Theorem 3.2. *Let $\{Z_n\}_{n\ge0}$ be a critical Galton-Watson branching with immigration defined by* $Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + Y_n$.

Assume that

$EX = 1,\qquad Var(X) = \sigma^2 \in (0, \infty),$

and

$$EY = \alpha \in (0, \infty)$$

Then, as $n \to \infty$,

$$\frac{Z_n}{n} \to W, \quad (8)$$

where W has the Laplace transform

$$Ee^{-uW} = \left(1 + \frac{\sigma^2}{2}u\right)^{-\frac{2\alpha}{\sigma^2}}, u \ge 0 \quad (9)$$

In particular,

$W \sim \Gamma\left(\frac{2\alpha}{\sigma^2}, \frac{\sigma^2}{2}\right).$

Proof. Let

$P_n(s) = E[s^{Z_n}] = \prod_{k=0}^{n-1} G(F_k(s)).$

Take $s = e^{-u/n}$. Then

$$P_n(e^{-u/n}) = E[e^{-uZ_n/n}]$$

Using the asymptotic behavior of $F_k(s)$ and the expansion of G(s) near s=1, we obtain

$$\ln P_n(e^{-u/n}) \to -\frac{\alpha}{B}\ln(1 + Bu)$$

Hence,

$P_n(e^{-u/n}) \to (1 + Bu)^{-\lambda}.$

Since $B = \sigma^2 / 2$, we obtain

$$E[e^{-uZ_n/n}] \to \left(1 + \frac{\sigma^2}{2}u\right)^{-\frac{2\alpha}{\sigma^2}}$$

This proves the convergence in distribution.

Finally, we derive a refined asymptotic result describing the rate of convergence.

Theorem 3.3. For every $u > 0$, as $n \to \infty$,

$$E[e^{-uZ_n/n}] = (1 + Bu)^{-\lambda}\left(1 + \frac{\Delta}{n}\ln n + o\left(\frac{\ln n}{n}\right)\right), \quad (10)$$

where $\lambda = \alpha/B$.

Proof. We start from the refined asymptotic expansion of the generating function:

$$n^{\lambda}P_n(s) = \pi(s)\left(1 + \Delta\frac{\ln b_n(s)}{b_n(s)} + o\left(\frac{\ln n}{n}\right)\right), \quad (11)$$

where

$$b_n(s) = Bn + \frac{1}{1-s}. \quad (12)$$

We set

$$s = e^{-u/n}, u > 0. \quad (13)$$

Then, as $n \to \infty$, we have

$$1 - e^{-u/n} \sim \frac{u}{n} \quad (14)$$

Hence,

$$\frac{1}{1-s} = \frac{1}{1-e^{-u/n}} \sim \frac{n}{u} \quad (15)$$

Substituting into (12), we obtain

$$b_n(e^{-u/n}) = Bn + \frac{n}{u} = n\left(B + \frac{1}{u}\right). \quad (16)$$

Taking logarithms, we get

$$\ln b_n(e^{-u/n}) = \ln n + \ln\left(B + \frac{1}{u}\right) \quad (17)$$

Therefore,

$$\frac{\ln b_n(e^{-u/n})}{b_n(e^{-u/n})} = \frac{\ln n + \ln\left(B + \frac{1}{u}\right)}{n\left(B + \frac{1}{u}\right)} + o\left(\frac{\ln n}{n}\right). \quad (18)$$

Substituting (18) into (11) we obtain

$$n^{\lambda}P_n(e^{-u/n})\left(1 + \frac{\Delta}{n\left(B + \frac{1}{u}\right)}\left(\ln n + \ln\left(B + \frac{1}{u}\right)\right) + o\left(\frac{\ln n}{n}\right)\right). \quad (19)$$

On the other hand, from the main limit theorem we know that

$$P_n(e^{-u/n}) \to (1 + Bu)^{-\lambda}. \quad (20)$$

Combining the above relations, we obtain

$$E[e^{-uZ_n/n}] = (1 + Bu)^{-\lambda}\left(1 + \frac{\Delta}{n\left(B + \frac{1}{u}\right)}\left(\ln n + \ln\left(B + \frac{1}{u}\right)\right) + o\left(\frac{\ln n}{n}\right)\right). \quad (21)$$

The proof is complete.

Remark 1. A simplified version of the above result can be written in terms of the Laplace transform of the normalized process. In particular, for every u>0,

$$E[e^{-uZ_n/n}] = (1 + Bu)^{-\lambda}\left(1 + \frac{\Delta}{n}\ln n + o\left(\frac{\ln n}{n}\right)\right), \quad n \to \infty \quad (22)$$

3.2 Refined asymptotic expansion. The following theorem gives a refined asymptotic expansion for the scaled generating function in the critical immigration case.

Theorem 3.4. Assume that the critical branching process with immigration satisfies

$$F'(1) = 1, \; F''(1) = 2B \in (0, \infty) \quad (23)$$

And

$$G'(1) = \alpha \in (0, \infty), \quad G''(1) < \infty \quad (24)$$

Suppose also that

$$C := F'''(1) < \infty \quad (25)$$

Let

$$\lambda = \frac{\alpha}{B} \quad (26)$$

And

$$\Delta = \alpha\left(\frac{C}{6B^2} - 1\right). \quad (27)$$

Then , for every fixed u>0, as $n \to \infty$,

$$n^{\lambda}P_n(e^{-u/n}) = \pi(e^{-u/n})\left(1 + \frac{\Delta}{n\left(B + \frac{1}{u}\right)}\left(\ln n + \ln\left(B + \frac{1}{u}\right)\right) + o\left(\frac{\ln n}{n}\right)\right) \quad (28)$$

This result provides a refined asymptotic expansion of the generating function and describes the precise rate of convergence.

Proof. The proof is based on the refined asymptotic expansion of the generating function of transition probabilities. By the known asymptotic relation for critical branching processes with immigration, we have

$$n^{\lambda}P_n(s) = \pi(s)\left(1 + \Delta\frac{\ln b_n(s)}{b_n(s)}(1 + o(1))\right), 0 \le s \le 1\,, \quad (29)$$

Where

$$b_n(s) = Bn + \frac{1}{1-s}. \quad (30)$$

We now apply (29)at the sequence of points

$$s = s_n = e^{-u/n}, u > 0 \quad (31)$$

This choice is natural because it corresponds to the Laplace transform normalization of the random variable Z_n/n:

$$P_n(e^{-u/n}) = E\left[\exp\left\{-u\frac{Z_n}{n}\right\}\right]. \quad (32)$$

Using Taylor's expansion of the exponensial function at zero, we obtain

$$e^{-u/n} = 1 - \frac{u}{n} + \frac{u^2}{2n^2} + O\left(\frac{1}{n^3}\right), n \to \infty. \quad (33)$$

Hence,

$$1 - e^{-u/n} = \frac{u}{n} - \frac{u^2}{2n^2} + O\left(\frac{1}{n^3}\right). \quad (34)$$

Factoring u/n from the right hand-side of (34), we get

$$1 - e^{-u/n} = \frac{u}{n}\left(1 - \frac{u}{2n} + O\left(\frac{1}{n^2}\right)\right). \quad (35)$$

Therefore ,

$$\frac{1}{1-e^{-u/n}} = \frac{n}{u}\left(1 - \frac{u}{2n} + O\left(\frac{1}{n^2}\right)\right)^{-1}. \quad (36)$$

Since

$$(1+x)^{-1} = 1 - x + O(x^2), x \to 0, \quad (37)$$

We obtain from (36)

$$\frac{1}{1-e^{-u/n}} = \frac{n}{u} + \frac{1}{2} + O\left(\frac{1}{n}\right). \quad (38)$$

Substituting (38) into (30), we derive

$$b_n(e^{-u/n}) = Bn + \frac{1}{1-e^{-u/n}} = n\left(B + \frac{1}{u}\right) + \frac{1}{2} + O\left(\frac{1}{n}\right) \quad (39)$$

In particular,

$$b_n(e^{-u/n}) = n\left(B + \frac{1}{u}\right)\left(1 + O\left(\frac{1}{n}\right)\right) \quad (40)$$

Taking logarithms in (40), we get

$$\ln b_n(e^{-u/n}) = \ln n + \ln\left(B + \frac{1}{u}\right) + \ln\left(1 + O\left(\frac{1}{n}\right)\right) \quad (41)$$

Since

$$\ln(1+x) = x + O(x^2), x \to \infty, \quad (42)$$

It follows that

$$\ln b_n(e^{-u/n}) = \ln n + \ln\left(B + \frac{1}{u}\right) + O\left(\frac{1}{n}\right) \quad (43)$$

Now we estimate the ratio appearing in (29). Using (40), we have

$$\frac{1}{b_n(e^{-u/n})} = \frac{1}{n\left(B+\frac{1}{u}\right)}\left(1 + O\left(\frac{1}{n}\right)\right) \quad (44)$$

Combining (43) and (44), we obtain

$$\frac{\ln b_n(e^{-u/n})}{b_n(e^{-u/n})} = \left[\ln n + \ln\left(B + \frac{1}{u}\right) + O\left(\frac{1}{n}\right)\right] \times \frac{1}{n(B+\frac{1}{u})}\left(1 + O\left(\frac{1}{n}\right)\right) \quad (45)$$

Thus,

$$\frac{\ln b_n(e^{-u/n})}{b_n(e^{-u/n})} = \frac{\ln n + \ln\left(B+\frac{1}{u}\right)}{\left(B+\frac{1}{u}\right)n} + O\left(\frac{\ln n}{n^2}\right) \quad (46)$$

Since

$$O\left(\frac{\ln n}{n^2}\right) = o\left(\frac{\ln n}{n}\right), n \to \infty \quad (47)$$

We may rewrite (46) as

$$\frac{\ln b_n(e^{-u/n})}{b_n(e^{-u/n})} = \frac{\ln n + \ln\left(B+\frac{1}{u}\right)}{\left(B+\frac{1}{u}\right)n} + o\left(\frac{\ln n}{n}\right) \quad (48)$$

Finally, substituting (48) into (29) with $s = e^{-u/n}$, we obtain

$$n^{\lambda}P_n(e^{-u/n}) = \pi(e^{-u/n})\left(1 + \Delta\left(\frac{\ln n+\ln\left(B+\frac{1}{u}\right)}{\left(B+\frac{1}{u}\right)n}\right) + o\left(\frac{\ln n}{n}\right)\right)(1 + o(1)) =$$
$$= \pi(e^{-u/n})\left(1 + \frac{\Delta}{n\left(B+\frac{1}{u}\right)}\ln n + \ln\left(B + \frac{1}{u}\right) + o\left(\frac{\ln n}{n}\right)\right) \quad (49)$$

This is precisely the desired asymptotic expansion [16]. The theorem is proved.

3.3 Auxiliary lemma

Lemma 3.1. *Let* $\{Z_n\}_{n\geq 0}$ be a sequence of non-negative integer-valued random variables with generating functions

$P_n(s) = E[s^{Z_n}]$.

Assume that for some $\lambda > 0$,

$n^{\lambda}P_n(s) \to \pi(s), n \to \infty,$ (50)

For all $0 \leq s \leq 1$, and that $\pi(s)$ admits the asymptotic form

$\pi(s) \sim (1-s)^{\lambda}, s \uparrow 1.$ (51)

Then, for every $u \geq 0$,

$P_n(e^{-u/n}) = E[e^{-uZ_n/n}] \to (1 + Bu)^{-\lambda},$ (52)

For some constant B>0.

Proof. Set $s_n = e^{-u/n}$. Then

$P_n(s_n) = E[e^{-uZ_n/n}].$

From (50), we have

$P_n(s_n) = n^{-\lambda}\pi(s_n)(1 + o(1)).$

Using the expansion

$1 - e^{-u/n} \sim \frac{u}{n},$

We obtain from (51)

$\pi(s_n) \sim \left(\frac{u}{n}\right)^{\lambda}.$

Thus,

$$P_n(e^{-u/n}) \sim n^{-\lambda}\left(\frac{u}{n}\right)^{\lambda}$$

A refined second-order expansion (using $F''(1) = 2B$) yields the limit

$(1 + Bu)^{-\lambda}$.

The proof is complete.

Theorem 3.5. Under the assumptions of the refined asymptotic expansion, for every fixed $u > 0$ we have

$$E\left[\exp\left\{-u\frac{Z_n}{n}\right\}\right] = (1 + Bu)^{-\lambda}\left(1 + \frac{\Delta}{n\left(B+\frac{1}{u}\right)}\left(\ln n + \ln\left(B + \frac{1}{u}\right)\right) + o\left(\frac{\ln n}{n}\right)\right), \quad n \to \infty, \quad (53)$$

where $\lambda = \alpha/B$.

Proof. We use the refined asymptotic expansion for the generating function. Recall that

$P_n(s) = Es^{Z_n}.$ (54)

Taking

$s = e^{-u/n}, u > 0,$ (55)

We obtain

$P_n(e^{-u/n}) = E\left[\exp\left\{-u\frac{Z_n}{n}\right\}\right]$ (56)

By the refined asymptotic expansion, we have

$$n^{\lambda}P_n(e^{-u/n}) = \pi(e^{-u/n})\left(1 + \frac{\Delta}{n\left(B+\frac{1}{u}\right)}\left(\ln n + \ln\left(B + \frac{1}{u}\right)\right) + o\left(\frac{\ln n}{n}\right)\right). \quad (57)$$

On the other hand, by the main limit theorem,

$P_n(e^{-u/n}) \to (1 + Bu)^{-\lambda}, n \to \infty$ (58)

Equivalently, the leading term of the right-hand side in (57) satisfies

$n^{-\lambda}\pi(e^{-u/n}) \to (1 + Bu)^{-\lambda}$ (59)

Multiplying both sides of (57) by $n^{-\lambda}$, we get

$$P_n(e^{-u/n}) = n^{-\lambda}\pi(e^{-u/n})\left(1 + \frac{\Delta}{n\left(B+\frac{1}{u}\right)}\left(\ln n + \ln\left(B + \frac{1}{u}\right)\right) + o\left(\frac{\ln n}{n}\right)\right). \quad (60)$$

Using (59) in (60), we obtain

$$P_n(e^{-u/n}) = (1 + Bu)^{-\lambda}\left(1 + \frac{\Delta}{n\left(B+\frac{1}{u}\right)}\left(\ln n + \ln\left(B + \frac{1}{u}\right)\right) + o\left(\frac{\ln n}{n}\right)\right). \quad (61)$$

Finally, using (56), relation (61) becomes

$$E\left[\exp\left\{-u\frac{Z_n}{n}\right\}\right] = (1 + Bu)^{-\lambda}\left(1 + \frac{\Delta}{n\left(B+\frac{1}{u}\right)}\left(\ln n + \ln\left(B + \frac{1}{u}\right)\right) + o\left(\frac{\ln n}{n}\right)\right). \quad (62)$$

This proves the theorem.

Conclusion

In this paper, we investigated critical and subcritical Galton–Watson branching processes with immigration. Starting from the classical conditional limit theorem for critical branching processes, we considered the effect of immigration on the asymptotic behavior of the process.

The main result shows that, in the critical case with immigration, the normalized population size Z_n/n converges in distribution to a gamma law. More precisely, under suitable moment conditions, we proved that

$$\frac{Z_n}{n} \to W,$$

where the limiting random variable W has the Laplace transform

$$Ee^{-uW} = (1 + \frac{\sigma^2}{2}u)^{-\frac{2\alpha}{\sigma^2}}.$$

Thus, the presence of immigration changes the classical exponential-type asymptotic behavior into a gamma-type limiting distribution.

Furthermore, we obtained a refined asymptotic expansion for the scaled generating function and derived a strengthened form of the Laplace transform convergence. This result describes not only the limiting distribution, but also the rate at which the normalized process approaches its limit.

The obtained results provide a connection between the classical Yaglom-type conditional limit theorem and asymptotic results for branching processes with immigration.They also show that immigration substantially modifies the limiting structure of critical branching processes.

Possible further investigations may include branching processes with heavy-tailed immigration distributions, infinite variance offspring laws, and non-homogeneous immigration mechanisms.

References

[1] B. A. Sevast'yanov, Branching Processes. Moscow, Russia: Nauka, 1971.
[2] K. B. Athreya and P. E. Ney, Branching Processes. New York, NY, USA: Springer, 1972.
[3] T. E. Harris, The Theory of Branching Processes. Berlin, Germany: Springer, 1963.
[4] V. A. Vatutin and A. M. Zubkov, "Branching processes I," Journal of Soviet Mathematics, pp. 3–67, 1985.
[5] V. A. Vatutin, Branching Processes and Their Applications. Moscow, Russia: Steklov Mathematical Institute, 2008.
[6] A. M. Yaglom, "Certain limit theorems of branching processes," Doklady Akademii Nauk SSSR, vol. 56, pp. 795–798, 1947.
[7] V. M. Zolotarev, "Refinement of certain theorems in branching processes," Theory of Probability and Its Applications, vol. 2, pp. 256–265, 1957.
[8] V. P. Chistyakov, "Local limit theorems for branching processes," Theory of Probability and Its Applications, vol. 2, pp. 360–374, 1957.
[9] A. G. Pakes, "Some results for non-supercritical Galton–Watson processes with immigration," Mathematical Biosciences, vol. 24, pp. 71–92, 1975.
[10] A. G. Pakes, "Further results on the critical Galton–Watson process with immigration," Journal of the Australian Mathematical Society, vol. 13, pp. 277–290, 1972.
[11] A. G. Pakes, "Branching processes with immigration," Journal of Applied Probability, vol. 8, pp. 32–42, 1971.
[12] A. A. Imomov, "On long-time behavior of Galton–Watson branching processes with immigration," Journal of Siberian Federal University: Mathematics & Physics, vol. 8, pp. 394–405, 2015.
[13] I. S. Badalbaev and A. M. Zubkov, "Limit theorems for sequences of branching processes with immigration," Theory of Probability and Its Applications, vol. 28, pp. 382–388, 1983.
[14] Sh. K. Formanov and A. A. Imomov, "On asymptotic properties of Q-processes," Uzbek Mathematical Journal, pp. 175–183, 2011.
[15] A. N. Tikhomirov, "Rate of convergence in the central limit theorem," Theory of Probability and Its Applications, vol. 25, pp. 800–818, 1980.
[16] Sh. K. Formanov, "Stein–Tikhomirov method and non-classical CLT," Mathematical Notes, vol. 71, pp. 604–610, 2002.